Stacks

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<u>Goal</u>: Construct moduli space for VB of rk n,
ie "space" BGl_n sth $\forall$ scheme T, $\quad Mor(T, BGl_n) \simeq \{\text{vbdles of rk } n \text{ on } T\}/_{\cong}$

We'd then get universal bdle $\mathcal{E}_{univ}$ on BGl_n ($\longleftrightarrow id \in Mor(BGl_n, BGl_n)$)
sth $\forall$ vbdle $\mathcal{E}$ of rk n on T, $\exists! \varphi_{\mathcal{E}} : T \to BGl_n$ sth $\varphi_{\mathcal{E}}^*(\mathcal{E}_{univ}) \simeq \mathcal{E}$.

<u>Problem</u>: BGl_n doesn't exist as a scheme (or algebraic space)!
(crucial pt: vbdles are locally trivial ...)

<u>Two solutions</u>: 1) Coarse moduli spaces
2) Keep the isomorphisms

<u>Ad 1)</u> A coarse moduli space for vbdles would be an alg. space sth
$\forall$ schemes T, $\exists$ morphism $\quad \{\text{vbdles of rk } n \text{ on } T\}/_{\cong} \longrightarrow Mor(T, BGl_n)$
(functorial but not iso)
$\downarrow$
[in general, no universal bdle!]

which is an iso for $T = pt$.

[A coarse moduli space exists if we consider stable vbdles]

<u>Ad 2)</u> Stacks $\rightsquigarrow$ iso's are part of the datum!

<u>Def.</u> A <u>stack</u> is a sheaf of groupoids $\mathcal{M}$: Schemes $\longrightarrow$ Groupoids $\subset$ Categories,
ie. (1) $\forall\ T \in$ Schemes, a groupoid $\mathcal{M}(T)$
(2) $\forall\ f: X \to Y$ in Schemes, a functor $f^*: \mathcal{M}(Y) \to \mathcal{M}(X)$
(3) $\forall\ X \xrightarrow{f} Y \xrightarrow{g} Z$, natural trafos $\varphi_{f,g}: f^* \circ g^* \Rightarrow (g \circ f)^*$
associative, in particular $\varphi_{id,g} = id$, $\varphi_{f,id} = id$,

with glueing conditions:

(1) (Objects) $\forall$ coverings $U_i \twoheadrightarrow T$ & $\mathcal{E}_i \in \mathcal{M}(U_i)$,

& $\forall\ \varphi_{ij} : \mathcal{E}_i|_{U_i \times_T U_j} \xrightarrow{\sim} \mathcal{E}_j|_{U_i \times_T U_j}$ with cocycle condition,

$\exists\ \mathcal{E} \in \mathcal{M}(T)$, unique up to iso,

with $\psi_i : \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{E}_i$ sth $\varphi_{ij} = \psi_j \circ \psi_i^{-1}$.

(2) (Morphisms) $\forall$ coverings $U_i \twoheadrightarrow T$ & $\mathcal{E}, \mathcal{F} \in \mathcal{M}(T)$

& $\forall\ \varphi_i : \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{F}|_{U_i}$ sth $\varphi_i|_{U_i \times_T U_j} = \varphi_j|_{U_i \times_T U_j}$,

$\exists!\ \varphi : \mathcal{E} \to \mathcal{F}$ with $\varphi|_{U_i} = \varphi_i$.

<u>Examples.</u> (1) C smooth proj curve, $n \in \mathbb{N}$

$\rightsquigarrow$ moduli stack Bun_n of rk n vbdles on C:

$\mathrm{Bun}_n(T) := \langle$ VB of rk n on $C \times T \rangle$

with iso's as morphisms (of VB's)

(Glueing conditions hold by descent theory)

(2) G affine alg gp

$\rightsquigarrow$ Classifying stack BG:

$BG(T) := \langle$ ppal G-bdles on $T \rangle$

(3) X scheme

$\rightsquigarrow$ stack $\underline{X}$ with $\underline{X}(T) := \mathrm{Mor}(T, X)$

(with only identity morphisms)

A stack isomorphic to some $\underline{X}$ is called <u>representable</u>.

<u>Lemma</u> (Yoneda for stacks) $\mathcal{M}$ stack, X scheme

$$\Rightarrow \quad \mathrm{Mor}_{\text{stacks}}(\underline{X}, \mathcal{M}) \simeq \mathcal{M}(X)$$

$$(F : \underline{X} \to \mathcal{M}) \longmapsto F(\mathrm{id}_X).$$

Consequence: $\mathcal{X}$ is determined by $\underline{X}$

$\rightsquigarrow$ write X instead of $\underline{X}$

Fibre products

$\mathcal{M}$ stack, Y, X schemes, $x : X \to \mathcal{M}$, $y : Y \to \mathcal{M}$

$$\rightsquigarrow (X \times_{\mathcal{M}} Y)(T) := \left\langle \begin{tikzcd} T \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "x"] \\ Y \arrow[r, "y"'] & \mathcal{M} \end{tikzcd} \text{ together with } \varphi : f^* x \xrightarrow{\sim} g^* y \right\rangle$$

$$= \operatorname{Isom}(x, y)$$

Example.
$$\begin{tikzcd} & pt \arrow[d, "triv"] \\ X \arrow[r] & BG \end{tikzcd}$$

$$\rightsquigarrow (X \times_{BG} pt)(T) = \left\langle \begin{tikzcd} T \arrow[r, "p"] \arrow[d, "f"'] & pt \arrow[d, "triv"] \\ X \arrow[r, "F_{\mathcal{E}}"'] & BG \end{tikzcd} \ \& \ \varphi : \underbrace{triv \circ p}_{F_{T \times G}} \xrightarrow{\sim} \underbrace{F_{\mathcal{E}} \circ f}_{F_{f^*\mathcal{E}}} \right\rangle$$

$$= \langle f, \text{ section } s : T \to f^* \mathcal{E} \rangle$$

$$= \mathcal{E}(T).$$

$\Rightarrow$ $pt \to BG$ is universal G-bundle

Def. $f : \mathcal{M} \to \mathcal{N}$ is <u>representable</u> if $\forall \underset{\text{scheme}}{X} \to \mathcal{N}$, $X \times_{\mathcal{N}} \mathcal{M}$ is an algebraic space.

Def. A stack $\mathcal{M}$ is called <u>algebraic</u> if

(1) $X \to \mathcal{M}$ is representable $\forall$ maps $\underset{\text{scheme}}{X} \to \mathcal{M}$

(2) $\exists$ atlas, ie $\underset{\text{scheme}}{U} \xrightarrow{u} \mathcal{M}$ sth

$X \times_{\mathcal{M}} U \to X$ is smooth & surjective $\forall \underset{\text{scheme}}{X} \to \mathcal{M}$.

(3) $\operatorname{Isom}(u, u) = U \times_{\mathcal{M}} U \longrightarrow U \times U$ is quasicompact & separated.

Example. Bun_n is an algebraic stack.

Indeed $\mathcal{M}_N(X) := \left\langle \begin{array}{l} \mathcal{E} \text{ rk } n \text{ vbdle on } C \times X \text{ sth } \mathcal{E} \otimes \mathcal{O}(N) \\ \text{is globally generated, } H^1(C \times X, \mathcal{E} \otimes \mathcal{O}(N)) = 0, \\ \{s_i\} \text{ basis of } H^0(C \times X, \mathcal{E} \otimes \mathcal{O}(N)) \end{array} \right\rangle$ is representable, $\mathcal{M}_N = \mathrm{Mor}(-, \mathcal{U}_N)$,

$\rightsquigarrow$ atlas $\mathcal{U} := \coprod_N \mathcal{M}_N \twoheadrightarrow \mathrm{Bun}_n$.

Geometric properties of alg. stacks

Def. An alg. stack $\mathcal{M}$ has property P $\leftarrow$ a property that can be checked locally on a smooth cover,

if $\exists$ atlas $u: \mathcal{U} \to \mathcal{M}$ sth $\mathcal{U}$ has P.

eg. normal, reduced, locally of finite presentation, ...

Def. A representable morphism $F: \mathcal{M} \to \mathcal{N}$ of alg stacks has property P

if $\exists$ atlas $u: \mathcal{U} \to \mathcal{N}$ sth $\mathcal{U} \times_{\mathcal{N}} \mathcal{M} \to \mathcal{U}$ has P.

a property that can be checked after smooth b.ch, eg closed/open immersion, affine, finite ...

Example. $\mathrm{Bun}_n^{ss} \hookrightarrow \mathrm{Bun}_n$ is open. (ss: semistable vbdles)

Lemma. Let $\mathcal{M}$ be an alg. stack locally of fin. presentation / Spec k, k a field,

sth $\forall$ local Artin algebras A, ideal $I \trianglelefteq A$, $I^2 = 0$,

one can complete any diagram

$$\begin{tikzcd} \mathrm{Spec}(A/I) \arrow[r] \arrow[d, hook] & \mathcal{M} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "\exists"] & \mathrm{Spec}(k) \end{tikzcd}$$

then $\mathcal{M} \to \mathrm{Spec}(k)$ is smooth.

Application: Bun_n is smooth (lift vbdle on $C \times \mathrm{Spec}\, A/I$ to $C \times \mathrm{Spec}\, A$ ...)

Sheaves on alg. stacks

Def. A quasicoherent sheaf $\mathcal{F}$ on an alg. stack $\mathcal{M}$ consists of

(1) $\forall x: X \xrightarrow{\text{smooth}} \mathcal{M}$, X scheme, a quasicoherent sheaf $\mathcal{F}_{X,x}$ on X.

(2) $\forall$ $\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "v"'] & U \arrow[d, "u"] \\ & \mathcal{M} \end{tikzcd}$ (2-)(commutative diagram), an iso $\varphi: u \circ f^*(\mathcal{F}) \xrightarrow{\sim} v^*(\mathcal{F})$.

Example structure sheaf $\mathcal{O}_{\mathcal{M}}$, $\mathcal{O}_{\mathcal{M},U,u} := \mathcal{O}_U$.

Relation with coarse moduli spaces

Def $\mathcal{M}$ alg. stack. An alg. space M together with a map $p : \mathcal{M} \to M$ is called a coarse moduli space for $\mathcal{M}$ if

(1)
$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr] & X \\ & M \arrow[u, dashed, "\exists !"'] \end{tikzcd}$$
$\forall$ schemes X

(2) $\forall \bar{k}$ alg closed,
$$\mathcal{M}(\bar{k})/_{\cong} \xrightarrow{\sim} M(\bar{k}) \quad \text{iso.}$$

Example: Bun_n^{stable} has a coarse moduli space M^{stable}

Construction: $\exists X$ sth $Bun_n^{stable} = [X/Gl_n]$,

then $M^{stable} = [X/PGl_n]$.

This is a $\mathbb{G}_m$-gerbe:

Def A morphism $F : \mathcal{M} \to \mathcal{N}$ of ~~alg.~~ stacks is called a gerbe over $\mathcal{N}$ if

(1) F is locally surjective
$$\begin{tikzcd} & T' \arrow[d, "\exists \text{ cover}"] \arrow[ddl, dashed] \\ & T \arrow[d] \\ \mathcal{M} \arrow[r, "F"'] & \mathcal{N} \end{tikzcd}$$

(2) $\forall t_1, t_2 : T \to \mathcal{M}$ with $F(t_1) \simeq F(t_2)$,
$\exists$ cover $T' \to T$ sth $t_1|_{T'} \simeq t_2|_{T'}$.

F is called $\mathbb{G}_m$-gerbe if $\forall t : T \to \mathcal{M}$, $Aut(t : T \to \mathcal{M}/\mathcal{M}) \simeq \mathbb{G}_m(T)$.

Lemma. For a $\mathbb{G}_m$-gerbe $F : \mathcal{M} \to \mathcal{N}$ the following are equivalent:

(1) $F : \mathcal{M} \to \mathcal{N}$ has a section.

(2) $\mathcal{M} \xrightarrow{\sim} B\mathbb{G}_m \times \mathcal{N}$

(3) $\exists$ line bdle of wt 1 on $\mathcal{M}$.

terminology: "trivial gerbe"